SECTION 11.2: POWER SERIES

Having studied series of real numbers $\sum_{k}^{\infty} a_k$, we now explore series of functions.

In this class, we restrict ourselves to series of polynomials specifically series with terms of the form $c_k\,(x-a)^k$ for whole numbers, k. Note that in this context, we define $(x-a)^0 = 1$ for all values of x, even $x = a$. These series are called **power series** and are formally represented as

$$f(x) = \sum_{k=0}^{\infty} c_k\,(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \ldots$$

The number a in the above expansion is called the **center** of the series.

As a motivational example, consider the power series centered at 0 below:

$$\sum_{k=0}^{\infty} \frac{1}{3^k}\,(x-0)^k = \sum_{k=0}^{\infty} \left(\frac{x}{3}\right)^k = 1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \frac{x^4}{81} + \ldots$$

We recognize this series immediately as a geometric series with common ratio $r = \dfrac{x}{3}$. Hence,

$$\sum_{k=0}^{\infty} \left(\frac{x}{3}\right)^k = 1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \frac{x^4}{81} + \ldots = \frac{1}{1-\frac{x}{3}} = \frac{3}{3-x}$$

This series converges exactly when $|r| = \left|\dfrac{x}{3}\right| < 1$, or, said differently, $|x| < 3$.

This inequality produces what is called an **interval of convergence** $\{x : -3 < x < 3\}$, which using interval notation is written: $(-3, 3)$. Note the midpoint of the interval of convergence is the center of the series, $x = 0$.

It turns out that the center of the power series will always act as a midpoint of the interval of convergence (which is why its called the 'center.') We define the **radius of convergence** to be the distance from the center of convergence out to the endpoints of the interval of convergence.[1] In this case, the radius of convergence is 3.

EXAMPLE 1: Graph $f(x) = \dfrac{3}{3-x}$ along with a few partial sums S_n of $\sum_{k=0}^{\infty} \left(\dfrac{x}{3}\right)^k$ using a graphing utility.

What sorts of behavior do you notice as you increase n?

Now consider the series:

$$\sum_{k=1}^{\infty} \frac{(x-2)^{k-1}}{k} = 1 + \frac{x-2}{2} + \frac{(x-2)^2}{3} + \frac{(x-2)^3}{4} + \ldots$$

This series is not geometric (why?) but we can still investigate for which values of x the series converges (even if we cant determine what the power series converges to.)

[1] If youre thinking 'center' and 'radius' means there's a 'circle' somewhere, you are correct. There is a circle (more precisely, a disk) of convergence here . . . but it lies in the complex plane.

Our principal tool here is the ratio test. Identifying $a_k = \dfrac{(x-2)^{k-1}}{k}$, $a_{k+1} = \dfrac{(x-2)^k}{k+1}$, we get:

$$\lim_{k\to\infty} \left| \frac{(x-2)^k}{k+1} \cdot \frac{k}{(x-2)^{k-1}} \right| = \lim_{k\to\infty} \frac{|x-2|^k}{|x-2|^{k-1}} \cdot \frac{k}{k-1} = \lim_{k\to\infty} |x-2| \frac{k}{k-1}$$

Note that we're not concerned about $x = 2$ here since the series will (by definition) converge at $x = 2$.

Since $|x-2|$ is independent of k, we can factor out the $|x-2|$ as a 'constant' from the limit provided the limit of what remains exists. We have $\lim\limits_{k\to\infty} \dfrac{k}{k-1} = 1$, we are cleared to use limit properties:

$$\lim_{k\to\infty} |x-2| \frac{k}{k-1} = |x-2| \lim_{k\to\infty} \frac{k}{k-1} = |x-2| \cdot 1 = |x-2|$$

The ratio test says the series $\displaystyle\sum_{k=1}^{\infty} \frac{(x-2)^{k-1}}{k}$ converges absolutely provided $|x-2| < 1$ and diverges if $|x-2| > 1$.

Hence, $\displaystyle\sum_{k=1}^{\infty} \frac{(x-2)^{k-1}}{k}$ converges absolutely when $-1 < x-2 < 1$ or $(1, 3)$ and diverges on $(-\infty, 1) \cup (3, \infty)$.

What happens at $x = 1$ and $x = 3$? For these numbers, $|x-2| = 1$ and the ratio test fails. Hence, we need to study what happens at these two values of x separately using a test other than the ratio test.

Substituting $x = 1$ into the power series results in which is the alternating harmonic series which converges:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

When x=3, we get the harmonic series which diverges:

$$\sum_{k=1}^{\infty} \frac{1^{k-1}}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

Hence, the interval of convergence for $\displaystyle\sum_{k=1}^{\infty} \frac{(x-2)^{k-1}}{k}$ is $[1, 3)$.

At this stage, we have no tools available to us to determine if $\displaystyle\sum_{k=1}^{\infty} \frac{(x-2)^{k-1}}{k}$ is a series representation of a more 'familiar' function (and in general, we'll never know.)

STRATEGIES FOR FINDING THE INTERVAL OF CONVERGENCE FOR A POWER SERIES:

Given a power series $f(x) = \sum_{k}^{\infty} c_k (x-a)^k$.

- If the $f(x)$ is geometric with common ratio, r, the interval of convergence is found by solving $|r| < 1$.
- If the $f(x)$ is not geometric, use the ratio test with $a_k = c_k (x-a)^k$. If the resulting interval contains finite endpoints, check each one individually in $f(x)$ to see if the series converges or diverges at the endpoints.

  NOTE: Endpoint series will necessarily require analysis using a test other than the ratio test.

EXAMPLE 2: Find the center, radius, and interval of convergence of the following power series.

1. $\displaystyle\sum_{k=0}^{\infty} \frac{(x-1)^k}{2^{2k}}$

   Ans: center: $x = 1$; radius of convergence: 4; interval of convergence: $(-3, 5)$ (NOTE: series is geometric!)

2. $\displaystyle\sum_{k=1}^{\infty} \frac{(x+2)^{k-1}}{k\,3^k}$

   Ans: center: $x = -2$; radius of convergence: 3; interval of convergence: $[-5, 1)$

3. **(VIDEO)** $\displaystyle\sum_{k=3}^{\infty} \frac{(x-5)^k}{k^2}$

   Ans: center: $x = 5$; radius of convergence: 1; interval of convergence: $[4, 6]$

4. **(VIDEO)** $\displaystyle\sum_{k=0}^{\infty} \frac{2^{2k+1}(x+1)^k}{(2k)!}$

   Ans: center: $x = -1$; radius of convergence: ∞; interval of convergence: $(-\infty, \infty)$

5. **(VIDEO)** $\displaystyle\sum_{k=0}^{\infty} \frac{k!\,(x-2)^{2k-1}}{42^{2k}}$

   Ans: center: $x = 2$; radius of convergence: 0; interval of convergence: $[2, 2] = \{2\}$

6. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{4^k}$

   Ans: center: $x = 0$; radius of convergence: 2; interval of convergence: $(-2, 2)$ (NOTE: series is geometric!)

7. $\displaystyle\sum_{k=1}^{\infty} \frac{(x+3)^{2k+1}}{k\,5^k}$

   Ans: center: $x = -3$; radius of convergence: $\sqrt{5}$; interval of convergence: $(-3-\sqrt{5}, -3+\sqrt{5})$

HOMEWORK: Section 11.2: 9 - 39 odd.

OPERATIONS ON POWER SERIES

We know from the geometric series formula that

$$f(x) = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 + \dots = \frac{1}{1-x},$$

for $-1 < x < 1$. Suppose we wished to find a power series representation for $g(x) = \dfrac{3}{1+2x}$.

Rewriting $g(x)$ in terms of $f(x)$, we see

$$g(x) = \frac{3}{1+2x} = 3\,\frac{1}{1-(-2x)} = 3f(-2x)$$

We can modify the power series representation for $f(x)$ to obtain a power series representation for $g(x)$:

$$g(x) = 3f(-2x) = 3\sum_{k=0}^{\infty}(-2x)^k = \sum_{k=0}^{\infty} 3\,(-1)^k\,2^k\,x^k = 3 - 6x + 12x^2 - 24x^3 \dots$$

with associated interval of convergence found by solving $|-2x| < 1$ or $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$.

Graphing a few partial sums of this power series along with $g(x)$ convinces us that, at least in this case, these methods seem to work! In fact, these sorts of machinations work in general as outlined in the theorem below.

THEOREM: (OPERATIONS ON POWER SERIES)

Suppose $f(x) = \sum_{k=0}^{\infty} c_k\,x^k$ and $g(x) = \sum_{k=0}^{\infty} d_k\,x^k$ on a common interval of convergence I. For all x in I:

- If $a \neq 0$, $a\,f(x) = a\sum_{k=0}^{\infty} c_k\,x^k = \sum_{k=0}^{\infty} (a\,c_k)\;x^k$.

- $x^N\,f(x) = x^N\sum_{k=0}^{\infty} c_k\,x^k = \sum_{k=0}^{\infty} c_k\,x^{k+N}$ provided N is a natural number.

- $f(x) \pm g(x) = \sum_{k=0}^{\infty} c_k\,x^k \pm \sum_{k=0}^{\infty} d_k\,x^k = \sum_{k=0}^{\infty} (c_k \pm d_k)\;x^k$

- $f\left(b\,x^N\right) = \sum_{k=0}^{\infty} c_k\,\left(b\,x^N\right)^k = \sum_{k=0}^{\infty} c_k\,b^k\,x^{Nk}$ provided $b\,x^N$ is in I.

EXAMPLE 3: Use the fact that for $|x| < 1$, $\dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{\infty} x^k$ to find power series representations for the following.

Include the interval of convergence.

1. $f(x) = \dfrac{2}{1-3x}$

Ans: $f(x) = \displaystyle\sum_{k=0}^{\infty} 2 \cdot 3^k\, x^k$ for x in $\left(-\dfrac{1}{3}, \dfrac{1}{3}\right)$.

2. **(VIDEO)** $g(x) = \dfrac{x^2}{3-x}$

Ans: $g(x) = \displaystyle\sum_{k=0}^{\infty} \dfrac{x^{k+2}}{3^{k+1}}$ for x in $(-3, 3)$.

3. **(VIDEO)** $h(x) = \dfrac{1}{x^2+4}$

Ans: $h(x) = \displaystyle\sum_{k=0}^{\infty} \dfrac{(-1)^k\, x^{2k}}{4^{k+1}}$ for x in $(-2, 2)$.

EXAMPLE 4: (VIDEO) Use the geometric series sum formula to find power series representations for each of the following functions centered at $x = 1$. Include the interval of convergence.

1. $f(x) = \dfrac{1}{5-x}$

Ans: $f(x) = \dfrac{1}{4-(x-1)} = \displaystyle\sum_{k=0}^{\infty} \dfrac{(x-1)^k}{4^{k+1}}$ for x in $(-3, 5)$.

2. $g(x) = \dfrac{1}{3+2x-x^2}$

Ans: $g(x) = \dfrac{1}{4-(x-1)^2} = \displaystyle\sum_{k=0}^{\infty} \dfrac{(x-1)^{2k}}{4^{k+1}}$ for x in $(-1, 3)$.

THE CALCULUS OF POWER SERIES

Recall for a function f, $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$, provided the limit exists.

Since power series converge **absolutely** within the interval of convergence, we can perform all of the operations to compute a derivative (including limits!) by doing them **term by term**. This means that we may take derivatives term by term as well! That is, if

$$f(x) = \sum_{k=0}^{\infty} c_k\,(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \ldots$$

then we can take the derivative of f by taking the derivative of each term then summing the results:

$$f'(x) = c_1(x-a) + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^4 + \ldots = \sum_{k=1}^{\infty} k\,c_k\,(x-a)^{k-1}$$

More succinctly, we have:

$$D_x\left[\sum_{k=0}^{\infty} c_k\,(x-a)^k\right] = \sum_{k=0}^{\infty} D_x\left[c_k\,(x-a)^k\right] = \sum_{k=1}^{\infty} k\,c_k\,(x-a)^{k-1}$$

Since derivatives go term by term, so does integration. That is:

$$\int f(x)\,dx = c + c_0 x + c_1\,\frac{(x-a)^2}{2} + c_2\,\frac{(x-a)^3}{3} + c_3\,\frac{(x-a)^4}{4} + \ldots = c + \sum_{k=0}^{\infty} c_k\,\frac{(x-a)^{k+1}}{k+1}$$

or, more succinctly:

$$\int\left[\sum_{k=0}^{\infty} c_k\,(x-a)^k\right]\,dx = \sum_{k=0}^{\infty}\left[\int c_k\,(x-a)^k\,dx\right] = c + \sum_{k=0}^{\infty} c_k\,\frac{(x-a)^{k+1}}{k+1}$$

Using the above formulas for $f'(x)$ and $\int f(x)\,dx$ it isn't to hard to show:

RADIUS OF CONVERGENCE FOR DERIVATIVES AND INTEGRALS

Given a power series $f(x) = \sum_{k=0}^{\infty} c_k\,(x-a)^k$ with radius of convergence R.

- The power series $f'(x) = \sum_{k=1}^{\infty} k\,c_k\,(x-a)^{k-1}$ has radius of convergence R.

  NOTE: We may **lose** convergence at endpoints.

- The power series $\int f(x)\,dx = c + \sum_{k=0}^{\infty} c_k\,\dfrac{(x-a)^{k+1}}{k+1}$ has radius of convergence R.

  NOTE: We may **gain** convergence at endpoints.

EXAMPLE 5: Recall the interval of convergence of $f(x) = \sum_{k=1}^{\infty} \frac{(x+2)^{k-1}}{k\,3^k}$ is $[-5, 1)$.

Find a power series representation for $f'(x)$ and accompanying interval of convergence.

Ans: $f'(x) = \sum_{k=2}^{\infty} \frac{k-1}{k\,3^k}(x+2)^{k-2}$ with interval of convergence $(-5, 1)$.

EXAMPLE 6: Let $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

1. Find the interval of convergence for f. Ans: $(-\infty, \infty)$

2. Find a power series representation for $f'(x)$ and determine the associated interval of convergence.

Ans: $f'(x) = \sum_{k=1}^{\infty} \frac{k\,x^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$ with interval of convergence $(-\infty, \infty)$.

EXAMPLE 7: Find a power representation for $f(x) = \tan^{-1}(x)$ by following the steps outlined below:

1. Use the geometric series formula to find a power series representation for $\frac{1}{x^2+1}$.

   State the corresponding interval of convergence.

   Ans: $\frac{1}{x^2+1} = \sum_{k=0}^{\infty} (-1)^k\,x^{2k}$; interval of convergence: $(-1, 1)$

2. Integrate the series in #1 to find a series for $\tan^{-1}(x) + c$.

   State the corresponding interval of convergence.

   Ans: $\tan^{-1}(x) = c + \sum_{k=0}^{\infty} \frac{(-1)^k\,x^{2k+1}}{2k+1}$; interval of convergence: $[-1, 1]$

3. Use the fact that $\tan^{-1}(0) = 0$ to find c and then find a series for $\frac{\pi}{4} = \tan^{-1}(1)$.

Ans: $c = 0$; $\tan^{-1}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\,x^{2k+1}}{2k+1}$; $\frac{\pi}{4} = \tan^{-1}(1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - + \ldots$

EXAMPLE 8: (VIDEO) Find a power representation for $f(x) = \ln(x)$ by following the steps outlined below:

1. Use the geometric series formula to find a power series representation for $\dfrac{1}{x}$ centered at $x = 1$.

   State the corresponding interval of convergence.

   $$\text{Ans: } \frac{1}{x} = \frac{1}{1 - (-(x-1))} = \sum_{k=0}^{\infty} (-1)^k (x-1)^k; \text{ interval of convergence: } (0, 2)$$

2. Integrate the series in #1 to find a series for $\ln(x) + c$.

   State the corresponding interval of convergence.

   $$\text{Ans: } \ln(x) = c + \sum_{k=0}^{\infty} \frac{(-1)^k (x-1)^{k+1}}{k+1}; \text{ interval of convergence: } (0, 2]$$

3. Use the fact that $\ln(1) = 0$ to find c and then find a series for $\ln(2)$.

   $$\text{Ans: } c = 0;\ \ln(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x-1)^{k+1}}{k+1};\ \ln(2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - + \ldots$$

HOMEWORK: Section 11.2: 41 - 61 odd.

MATH 2600: EXTRA PRACTICE WITH POWER SERIES

For each function $f(x)$ given below:

- Find the center and radius of convergence.
- Find the interval of convergence.
- Find and simplify a formula for $f'(x)$ and find the interval of convergence for f'.
- Find a formula for $\int f(x)\,dx$ and find the interval of convergence for $\int f(x)\,dx$.

1. $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{(n+1)\,2^n}$
2. $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k\,(x+1)^k}{(2k-1)\,3^k}$
3. $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{2^n\,x^n}{3^{2n-1}}$
4. $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$

ANSWERS:

1.
- For $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{(n+1)\,2^n}$: Center: $a = 3$; Radius: $R = 2$; Interval of Convergence: $[1, 5)$.
- $f'(x) = \displaystyle\sum_{n=1}^{\infty} \frac{n\,(x-3)^{n-1}}{(n+1)\,2^n}$; Interval of Convergence: $(1, 5)$.
- $\displaystyle\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)^2\,2^n}$; Interval of Convergence: $[1, 5]$.

2.
- For $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k\,(x+1)^k}{(2k-1)\,3^k}$: Center: $a = -1$; Radius: $R = 3$; Interval of Convergence: $(-4, 2]$.
- $f'(x) = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k\,k\,(x+1)^{k-1}}{(2k-1)\,3^k}$; Interval of Convergence: $(-4, 2)$.
- $\displaystyle\int f(x)\,dx = C + \sum_{k=1}^{\infty} \frac{(-1)^k\,(x+1)^{k+1}}{(k+1)(2k-1)\,3^k}$; Interval of Convergence: $[-4, 2]$.

3.
- For $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{2^n\,x^n}{3^{2n-1}}$: Center: $a = 0$; Radius: $R = \dfrac{9}{2}$; Interval of Convergence: $\left(-\dfrac{9}{2}, \dfrac{9}{2}\right)$.
- $f'(x) = \displaystyle\sum_{n=1}^{\infty} \frac{2^n\,n\,x^{n-1}}{3^{2n-1}}$; Interval of Convergence: $\left(-\dfrac{9}{2}, \dfrac{9}{2}\right)$.
- $\displaystyle\int f(x)\,dx = C + \sum_{n=1}^{\infty} \frac{2^n\,x^{n+1}}{(n+1)\,3^{2n-1}}$; Interval of Convergence: $\left[-\dfrac{9}{2}, \dfrac{9}{2}\right)$.

4.
- For $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$: Center: $a = 0$; Radius: $R = \infty$; Interval of Convergence: $(-\infty, \infty)$.
- $f'(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k\,(2k+1)\,x^{2k}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k\,x^{2k}}{(2k)!}$; Interval of Convergence: $(-\infty, \infty)$.
- $\displaystyle\int f(x)\,dx = C + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{(2k+2)(2k+1)!} = C + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{(2k+2)!}$; Interval of Convergence: $(-\infty, \infty)$.

  NOTE: You can rewrite $C + \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{(2k+2)!} = C_1 + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$.